

# HOMOGENIZATION OF VISCOUS AND NON-VISCOUS HJ EQUATIONS: A REMARK AND AN APPLICATION

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**ABSTRACT.** It was pointed out by P.-L. Lions, G. Papanicolaou, and S.R.S. Varadhan in their seminal paper [25] that, for first order Hamilton-Jacobi (HJ) equations, homogenization starting with affine initial data implies homogenization for general uniformly continuous initial data. The argument makes use of some properties of the HJ semi-group, in particular, the finite speed of propagation. The last property is lost for viscous HJ equations. In this paper we prove the above mentioned implication in both viscous and non-viscous cases. Our proof relies on a variant of Evans's perturbed test function method. As an application, we show homogenization in the stationary ergodic setting for viscous and non-viscous HJ equations in one space dimension with non-convex Hamiltonians of specific form. The results are new in the viscous case.

## 1. INTRODUCTION

Consider a family of equations of the form

$$\partial_t u^\varepsilon - \varepsilon \operatorname{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 u^\varepsilon \right) + H \left( \frac{x}{\varepsilon}, D_x u^\varepsilon \right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d, \quad (\text{HJ}_\varepsilon)$$

where  $\varepsilon > 0$ ,  $A$  is a  $d \times d$  symmetric and positive semi-definite matrix with Lipschitz and bounded coefficients, and  $H$ , the Hamiltonian, is a continuous function on  $\mathbb{R}^d \times \mathbb{R}^d$ , coercive in the gradient variable, uniformly with respect to  $x$ . If  $A \not\equiv 0$ , we shall refer to  $(\text{HJ}_\varepsilon)$  as viscous Hamilton-Jacobi equation.

Under suitable assumptions on  $H$ , equation  $(\text{HJ}_\varepsilon)$  satisfies a comparison principle, yielding the existence of a unique viscosity solution  $u^\varepsilon$  (in a proper class of continuous functions) subject to the initial condition  $u^\varepsilon(0, \cdot) = g$  in  $\mathbb{R}^d$ , with  $g$  uniformly continuous in  $\mathbb{R}^d$ . We shall say that *the equation  $(\text{HJ}_\varepsilon)$  homogenizes* if there exists a continuous function  $\overline{H} : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $u^\varepsilon$  converges, locally uniformly in  $[0, +\infty) \times \mathbb{R}^d$  as  $\varepsilon \rightarrow 0^+$ , to the unique viscosity solution  $\overline{u}$  of the following *effective equation*

$$\partial_t \overline{u} + \overline{H}(D_x \overline{u}) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d \quad (1)$$

satisfying  $\overline{u}(0, \cdot) = g$  for every uniformly continuous function  $g$  on  $\mathbb{R}^d$ .

The study of homogenization of Hamilton-Jacobi equations was initiated by P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan around 1987. Their seminal paper [25] was concerned with homogenization of first order Hamilton-Jacobi equations in the periodic setting, i. e. when  $A \equiv 0$  and  $H(\cdot + z, \cdot) \equiv H(\cdot, \cdot)$  for all  $z \in \mathbb{Z}^d$ . In particular, Section I.2 of [25] explains why, for first order Hamilton-Jacobi equations, homogenization for linear initial data implies homogenization for general uniformly continuous initial data. The outline of the proof provided in [25] uses characterization results for strongly continuous

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semi-groups on  $\text{UC}([0, +\infty) \times \mathbb{R}^d)$ , see [23, 24], as well as a uniform (in  $\varepsilon$ ) speed of propagation for the semigroup generated by the Cauchy problem associated to  $(\text{HJ}_\varepsilon)$ , which holds true since  $A \equiv 0$ .

In this paper, we give a proof of this fact for both viscous and non-viscous Hamilton-Jacobi equations under a quite general set of assumptions on the equation  $(\text{HJ}_\varepsilon)$ . More precisely, we will prove the following result (Theorem 3.1): assume that

$$u_\theta^\varepsilon(t, x) \xrightarrow{\varepsilon \rightarrow 0^+} \langle \theta, x \rangle - t\overline{H}(\theta) \quad \text{locally uniformly in } [0, +\infty) \times \mathbb{R}^d \quad (2)$$

for every  $\theta \in \mathbb{R}^d$  and for some continuous and coercive function  $\overline{H} : \mathbb{R}^d \rightarrow \mathbb{R}$ , where  $u_\theta^\varepsilon$  is the solution to  $(\text{HJ}_\varepsilon)$  with initial datum  $u_\theta^\varepsilon(0, x) = \langle \theta, x \rangle$ .<sup>1</sup> Then equation  $(\text{HJ}_\varepsilon)$  homogenizes. Note that, if homogenization takes place, then the effective Hamiltonian is completely characterized in terms of the limit of  $u_\theta^\varepsilon(1, 0) = \varepsilon u_\theta(1/\varepsilon, 0)$  as  $\varepsilon \rightarrow 0^+$ , with equality holding due to the identity  $u_\theta^\varepsilon(t, x) = \varepsilon u_\theta(t/\varepsilon, x/\varepsilon)$  on  $[0, +\infty) \times \mathbb{R}^d$ . Our proof relies on a variant of the elegant and powerful perturbed test function method due to L. C. Evans [16], where the test function is perturbed by a term of the form  $\varepsilon v_\theta(t/\varepsilon, x/\varepsilon)$  with  $v_\theta(t, x) := u_\theta(t, x) - \langle \theta, x \rangle + t\overline{H}(\theta)$  for a proper  $\theta \in \mathbb{R}^d$ . We recall that the standard homogenization approach consists in choosing as  $v_\theta$  a (time-independent) sub-linear solution of the cell problem

$$-\text{tr} \left( A(x) D_x^2 v \right) + H(x, \theta + D_x v) = \overline{H}(\theta) \quad \text{in } \mathbb{R}^d,$$

also known as (*exact*) *corrector*. In the periodic setting, correctors always exist and are, moreover, periodic, but in more general settings sub-linear correctors need not exist as shown in [26]. Thus, loosely speaking,  $\{v_\theta(t, x) : \theta \in \mathbb{R}^d\}$  can be thought of as a family of  $t$ -dependent correctors.

Besides the beauty and simplicity of this abstract result *in se*, our main interest is motivated by applications to the stationary ergodic setting. In Section 4.1 we take a further step and show that, in order to have (2) with probability one, it suffices to check that  $\lim_{\varepsilon \rightarrow 0^+} u_\theta^\varepsilon(1, 0, \omega) = -\overline{H}(\theta)$  almost surely with respect to  $\omega$ . Our general results are applied in Section 4.2 to obtain homogenization for a one dimensional Hamilton-Jacobi equation of the form

$$u_t^\varepsilon - \varepsilon A\left(\frac{x}{\varepsilon}\right) u_{xx}^\varepsilon + H\left(\frac{x}{\varepsilon}, u_x^\varepsilon, \omega\right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}, \quad (\text{HJ}_\varepsilon^\omega)$$

where the stationary random field  $H : \Omega \rightarrow \mathbb{C}(\mathbb{R}^d \times \mathbb{R}^d)$  takes values in a special class of non-convex and uniformly superlinear Hamiltonians. More precisely, we will assume that  $H$  is *pinned* at finitely many values  $p_1 < p_2 < \dots < p_n$ , meaning that  $H(\cdot, p_i, \cdot)$  is constant on  $\mathbb{R} \times \Omega$  for each fixed  $i$  (see Definition 4.8), and *piecewise convex* in  $p$ , meaning that  $H(x, \cdot, \omega)$  is convex on each of the intervals  $(-\infty, p_1)$ ,  $(p_1, p_2)$ ,  $\dots$ ,  $(p_{n-1}, +\infty)$ , for every fixed  $(x, \omega)$ , see Theorem 4.10. When  $A \equiv 0$ , we can weaken the convexity assumption to level-set convexity.

In order to obtain this result, we consider first the case of a stationary Hamiltonian which is pinned at  $p = 0$ . Such a Hamiltonian can be always written as

$$H(x, p, \omega) = \min\{H_-(x, p, \omega), H_+(x, p, \omega)\} = \begin{cases} H_-(x, p, \omega) & \text{if } p \leq 0 \\ H_+(x, p, \omega) & \text{if } p \geq 0, \end{cases} \quad (3)$$

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<sup>1</sup>When  $\varepsilon = 1$ , we shall simply write  $u_\theta$  in place of  $u_\theta^1$ .

where  $H_{\pm} : \Omega \rightarrow C(\mathbb{R}^d \times \mathbb{R}^d)$  are stationary random fields, uniformly coercive in  $p$ , and satisfying  $H_{\pm}(\cdot, 0, \cdot) \equiv h_0$  on  $\mathbb{R} \times \Omega$  for some constant  $h_0 \in \mathbb{R}$ . The core of our argument consists in showing that, for every fixed  $\omega$ , the function  $u_{\theta}^{\varepsilon}$  enjoys the same kind of monotonicity as its initial datum  $\theta x$  with respect to the  $x$  variable. In particular,  $u_{\theta}^{\varepsilon}$  is also a solution of

$$u_t^{\varepsilon} - \varepsilon A\left(\frac{x}{\varepsilon}\right) u_{xx}^{\varepsilon} + H_{\pm}\left(\frac{x}{\varepsilon}, u_x^{\varepsilon}, \omega\right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}, \quad (4)$$

according to the sign of  $\theta$ . If we assume, in addition, that the equation (4) homogenizes for both  $H_+$  and  $H_-$ , then we immediately conclude that  $(HJ_{\varepsilon}^{\omega})$  homogenizes for all linear initial data  $g(x) = \theta x$ . An application of our previous results gives homogenization of  $(HJ_{\varepsilon}^{\omega})$  for general uniformly continuous initial data. Furthermore, if we denote by  $\overline{H}_{\pm}$  the corresponding effective Hamiltonians associated to  $H_{\pm}$ , respectively, then the effective Hamiltonian  $\overline{H}$  can be expressed by the following formula

$$\overline{H}(\theta) = \min\{\overline{H}_-(\theta), \overline{H}_+(\theta)\} \quad \text{for every } \theta \in \mathbb{R}.$$

The assumption that (4) homogenizes is, for instance, fulfilled when  $H_{\pm}$  are convex, or even level-set convex when  $A \equiv 0$ , in view of known homogenization results [6, 13, 22, 27–29]. In this case, we conclude that equation  $(HJ_{\varepsilon}^{\omega})$  homogenizes when  $H$  is of the form (3). In particular, we infer that  $\overline{H}$  can be neither convex nor even level-set convex, see Remark 4.4.

The case when  $H$  is pinned at  $p_0 \neq 0$  can be always reduced to the one considered above by replacing  $H$  with  $H(\cdot, p_0 + \cdot, \cdot)$ , see Remark 4.9. The extension of the homogenization result to piecewise convex stationary Hamiltonians with multiple pinning points is obtained by induction on the number of pinning points, see Theorem 4.10. The basic idea is that a piecewise convex stationary Hamiltonian with  $n$  pinning points can be always written in the form (3) for some  $H_{\pm}$  of same type but with fewer pinning points.

Although we are able to treat only a special family of Hamiltonians in one dimension, the results are new in the viscous case. We stress that the Hamiltonians we consider are typically neither level-set convex nor satisfy any homogeneity condition with respect to  $p$ , and are, thus, not covered by examples treated in [17] or [3, equation (1.6)]. Even though the results of [3] hold in all dimensions, they require a finite range dependence condition on the coefficients. The last assumption is typically considered to be very restrictive but, as it was recently demonstrated in [30], homogenization in general stationary ergodic settings and dimensions larger than one need not hold for non-convex but otherwise “standard” Hamiltonians without some condition on the decay of correlations of the coefficients. Earlier works on non-convex homogenization in the stationary ergodic setting include homogenization for level-set convex Hamiltonians in the non-viscous case in one space dimension [13] and in any dimension [5], see also [17] for some additional results and extensions to viscous case. The first example of homogenization for a class of Hamiltonians which are not level-set convex was given in [7] for the non-viscous case in all dimensions. Papers [8] and [19] provide quite general non-convex homogenization results for one-dimensional non-viscous HJ equations.

We end this introduction by comparing our condition (2) with a notion of ergodicity introduced in [2] in the context of periodic homogenization. Let us set  $F(x, p, X) := -\text{tr}(A(x)X) + H(x, p)$  and assume that  $F$  is  $\mathbb{Z}^d$ -periodic in  $x$ . Following [2], the function  $F$  is said to be *ergodic at*  $\theta \in \mathbb{R}^d$  if the periodic solution  $w_{\theta}$  of

$$w_t - \text{tr}(A(y)D_x^2 w) + H(x, \theta + D_x w) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d$$

with initial condition  $w(0, \cdot) = 0$  on  $\mathbb{R}^d$  satisfies  $w_\theta(t, x)/t \rightarrow c$  as  $t \rightarrow +\infty$  uniformly in  $x$ , where  $c = c(\theta)$  is a constant. It was shown in [2, Section 2.5] that the ergodicity of  $F$  at each  $\theta \in \mathbb{R}^d$  implies that  $(\text{HJ}_\varepsilon)$  homogenizes, with  $\overline{H}(\theta) := -c(\theta)$  for every  $\theta \in \mathbb{R}^d$ . This holds, of course, under proper assumptions on the parabolic equation associated with  $F$ , that are for instance fulfilled when  $A$  and  $H$  satisfy our standing assumptions (A1)-(A2) and (H1)-(H2), respectively, and the Cauchy problem associated to  $(\text{HJ}_\varepsilon)$  is well-posed in a suitable class of continuous functions (see Section 2 for more details). To see a connection with our results, observe that the above notion of ergodicity can be thought of as a version of homogenization of  $(\text{HJ}_\varepsilon)$  for linear initial data. Indeed, note that  $u_\theta^\varepsilon(t, x) = \langle \theta, x \rangle + \varepsilon w_\theta(t/\varepsilon, x/\varepsilon)$ . The ergodicity is equivalent to the statement that, for every fixed  $t > 0$ ,

$$\lim_{\varepsilon \rightarrow 0+} u_\theta^\varepsilon(t, x) = \langle \theta, x \rangle - t \overline{H}(\theta) \quad \text{uniformly in } x \in \mathbb{R}^d. \quad (5)$$

Thus, the above cited result from [2] can be restated as follows: if the convergence (5) takes place for every fixed  $t > 0$  and  $\theta \in \mathbb{R}^d$ , then  $(\text{HJ}_\varepsilon)$  homogenizes. Taking into account that we do not assume that  $A$  and  $H$  are periodic in  $x$  and, thus, forgo the advantages of having  $x$  in a compact set, it is clear that our Theorem 3.1 is a close cousin of the quoted result from [2].

The paper is organized as follows. Section 2 is devoted to preliminaries: basic notation, definitions, comparison principles, existence and properties of solutions to  $(\text{HJ}_\varepsilon)$ , both in the viscous and in the non-viscous case. Section 3 contains our first results concerning the connection between homogenization and homogenization with linear initial data. In Section 4 we introduce the stationary ergodic formulation, present a stationary ergodic version of Theorem 3.1, and use blue our general results to show homogenization for a class of non-convex viscous HJ equations in one space dimension.

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## 2. PRELIMINARIES

Throughout the paper, we denote by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$ , respectively, the scalar product and the Euclidean norm on  $\mathbb{R}^d$ ,  $d \in \mathbb{N}$ . We let  $B_R(x_0)$  and  $B_R$  be the open balls in  $\mathbb{R}^d$  of radius  $R$  centered at  $x_0$  and 0, respectively. For a given subset  $E$  of  $\mathbb{R}^d$ , we will denote by  $\overline{E}$  its closure.

By modulus of continuity we mean a nondecreasing function from  $[0, +\infty)$  to  $[0, +\infty)$ , vanishing and continuous at 0.

Given a metric space  $X$ , we write  $\varphi_n \rightrightarrows_{\text{loc}} \varphi$  on  $X$  when the sequence of functions  $(\varphi_n)_n$  uniformly converges to  $\varphi$  on compact subsets of  $X$ . We denote by  $\text{Lip}(X)$ ,  $\text{UC}(X)$ ,  $\text{LSC}(X)$ , and  $\text{USC}(X)$  the space of Lipschitz continuous, uniformly continuous, lower

semicontinuous, and upper semicontinuous real functions on the metric space  $X$ , respectively.

Given an open subset  $U$  of either  $\mathbb{R}^d$  or  $\mathbb{R}^{d+1}$  and a measurable function  $g : U \rightarrow \mathbb{R}$ , we write  $\|g\|_{L^\infty(U)}$ , or simply  $\|g\|_\infty$  when no ambiguity is possible, to refer to the usual  $L^\infty$ -norm of  $g$ . The space of essentially bounded functions on  $U$  is denoted by  $L^\infty(U)$ .

Let  $k \in \mathbb{N}$ . We denote by  $C^k(\mathbb{R}^d)$  the space of continuous functions that are differentiable in  $\mathbb{R}^d$  with continuous derivatives up to order  $k$  inclusively, and set  $C^\infty(\mathbb{R}^d) := \bigcap_{k \in \mathbb{N}} C^k(\mathbb{R}^d)$ . Furthermore, for  $k \geq 2$ , we denote by  $\text{Lip}^k(\mathbb{R}^d)$  the space of Lipschitz functions defined on  $\mathbb{R}^d$  that have Lipschitz derivatives up to order  $k - 1$  inclusively. In the sequel we shall often use the notation

$$\|D_x g\|_\infty := \sum_{i=1}^d \|\partial_{x_i} g\|_\infty, \quad \|D_x^2 g\|_\infty := \sum_{i,j=1}^d \|\partial_{x_i x_j}^2 g\|_\infty.$$

We shall record the following basic density result for future use.

**Lemma 2.1.** *Let  $k \in \mathbb{N}$ . The space of functions  $C^\infty(\mathbb{R}^d) \cap \text{Lip}^k(\mathbb{R}^d)$  is dense in  $\text{UC}(\mathbb{R}^d)$  with respect to the  $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$  norm.*

*Proof.* Since  $\text{Lip}(\mathbb{R}^d)$  is dense in  $\text{UC}(\mathbb{R}^d)$ , see for instance [20, Theorem 1], it is enough to show that any  $g \in \text{Lip}(\mathbb{R}^d)$  can be uniformly approximated in  $\mathbb{R}^d$  by functions in  $C^\infty(\mathbb{R}^d) \cap \text{Lip}^k(\mathbb{R}^d)$ . But this readily follows by regularizing  $g$  via a convolution with a standard mollification kernel.  $\square$

Throughout the paper, we denote by  $A(x)$  a positive semi-definite symmetric  $d \times d$  matrix, depending on  $x \in \mathbb{R}^d$ , with bounded and Lipschitz square root, namely,  $A = \sigma^T \sigma$  for some  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{m \times d}$ , where  $\sigma$  satisfies the following conditions: there is a constant  $\Lambda_A > 0$  such that

- (A1)  $|\sigma(x)| \leq \Lambda_A$  for every  $x \in \mathbb{R}^d$ ;
- (A2)  $|\sigma(x) - \sigma(y)| \leq \Lambda_A |x - y|$  for every  $x, y \in \mathbb{R}^d$ .

We stress that the case  $A \equiv 0$  is included. We let  $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuous function, hereafter called *Hamiltonian*, satisfying the following assumptions:

- (H1)  $H \in \text{UC}(\mathbb{R}^d \times B_R)$  for every  $R > 0$ ;
- (H2) there exist two continuous, coercive, and nondecreasing functions  $\alpha, \beta : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that

$$\alpha(|p|) \leq H(x, p) \leq \beta(|p|) \quad \text{for every } (x, p) \in \mathbb{R}^d \times \mathbb{R}^d.$$

By coercive we mean that  $\lim_{R \rightarrow +\infty} \alpha(R) = \lim_{R \rightarrow +\infty} \beta(R) = +\infty$ .

Assumption (H2) amounts to saying that the Hamiltonian is coercive and locally bounded in  $p$ , uniformly with respect to  $x$ .

A Hamiltonian  $H$  will be termed *convex* if  $H(x, \cdot)$  is convex on  $\mathbb{R}^d$  for every  $x \in \mathbb{R}^d$ , and non-convex otherwise.

Let us consider the unscaled equation

$$\partial_t u - \text{tr}(A(x) D_x^2 u) + H(x, D_x u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d. \quad (\text{HJ}_1)$$

We shall say that a function  $v \in \text{USC}((0, +\infty) \times \mathbb{R}^d)$  is an (upper semicontinuous) *viscosity subsolution* of (HJ<sub>1</sub>) if, for every  $\phi \in C^2((0, +\infty) \times \mathbb{R}^d)$  such that  $v - \phi$  attains

a local maximum at  $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^d$ , we have

$$\partial_t \phi(t_0, x_0) - \operatorname{tr}(A(x_0) D_x^2 \phi(t_0, x_0)) + H(x_0, D_x \phi(t_0, x_0)) \leq 0. \quad (2.1)$$

Any such test function  $\phi$  will be called *supertangent* to  $v$  at  $(t_0, x_0)$ .

We shall say that  $w \in \text{LSC}((0, +\infty) \times \mathbb{R}^d)$  is a (lower semicontinuous) *viscosity subsolution* of  $(\text{HJ}_1)$  if, for every  $\phi \in C^2((0, +\infty) \times \mathbb{R}^d)$  such that  $w - \phi$  attains a local minimum at  $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^d$ , we have

$$\partial_t \phi(t_0, x_0) - \operatorname{tr}(A(x_0) D_x^2 \phi(t_0, x_0)) + H(x_0, D_x \phi(t_0, x_0)) \geq 0. \quad (2.2)$$

Any such test function  $\phi$  will be called *subtangent* to  $w$  at  $(t_0, x_0)$ . A continuous function on  $(0, +\infty) \times \mathbb{R}^d$  is a *viscosity solution* of  $(\text{HJ}_1)$  if it is both a viscosity sub and supersolution. Solutions, subsolutions, and supersolutions will be always intended in the viscosity sense, hence the term *viscosity* will be omitted in the sequel.

It is well known, see for instance [9], that the notions of viscosity sub and supersolutions are local, in the sense that the test function  $\phi$  needs to be defined only in a neighborhood of the point  $(t_0, x_0)$ . Moreover, up to adding to  $\phi$  a superquadratic term, such a point can be always assumed to be either a strict local maximum or a strict local minimum point of  $u - \phi$ . If this case we shall say that  $\phi$  is a *strict supertangent* (resp., *strict subtangent*) to  $u$  at  $(t_0, x_0)$ .

We shall denote by  $\mathcal{K}$  the space of functions  $u : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$  for which there exists a function  $f \in \text{UC}(\mathbb{R}^d)$ , depending on  $u$ , such that, for every  $T > 0$ ,

$$|u(t, x) - f(x)| \leq C_T \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d \quad (2.3)$$

for some constant  $C_T > 0$ . We let  $\mathcal{K}_*$  be the subspace of functions  $u \in \mathcal{K}$  which satisfy the following uniform continuity condition in time at  $t = 0$ :

for every  $a > 0$  there exists  $M_a > 0$  such that

$$|u(t, x) - u(0, x)| \leq a + t M_a \quad \text{for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d. \quad (*)$$

**Definition 2.2.** We shall say that the Cauchy problem for  $(\text{HJ}_1)$  is  $\mathcal{K}_*$ -*well-posed* if the following two properties hold:

- (a) (Existence) for every  $g \in \text{UC}(\mathbb{R}^d)$ , there exists a continuous function  $u \in \mathcal{K}_*$  which solves  $(\text{HJ}_1)$  and satisfies the initial condition  $u(0, \cdot) = g$  on  $\mathbb{R}^d$ ;
- (b) (Comparison) if  $u_1, u_2$  are continuous solutions to  $(\text{HJ}_1)$  belonging to  $\mathcal{K}_*$  with  $u_1(0, \cdot), u_2(0, \cdot) \in \text{UC}(\mathbb{R}^d)$ , then

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_1(0, \cdot) - u_2(0, \cdot)\|_{L^\infty(\mathbb{R}^d)} \quad \text{for all } t \in [0, +\infty).$$

We start by proving a comparison principle, which will be used several times throughout the paper.

**Proposition 2.3.** Assume that  $A$  satisfies (A1)-(A2) and  $H$  satisfies (H1). Let  $v \in \text{USC}([0, +\infty) \times \mathbb{R}^d)$  and  $w \in \text{LSC}([0, +\infty) \times \mathbb{R}^d)$  be, respectively, a sub and a supersolution of  $(\text{HJ}_1)$  belonging to  $\mathcal{K}$ . Let us furthermore assume that, for every  $T > 0$ , either  $D_x v$  or  $D_x w$  belongs to  $(L^\infty([0, T] \times \mathbb{R}^d))^d$ . Then

$$v(t, x) - w(t, x) \leq \sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)) \quad \text{for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$



*Proof.* Excluding trivial cases and adding a constant to  $w$  as necessary, we can assume that  $\sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)) = 0$ . Our task is thus reduced to proving that  $v \leq w$  on  $[0, +\infty) \times \mathbb{R}^d$ . By definition of  $\mathcal{K}$  and Lemma 2.1, there exists a function  $g \in C^\infty(\mathbb{R}^d) \cap \text{Lip}^3(\mathbb{R}^d)$  such that  $v - g$  is bounded in  $[0, T] \times \mathbb{R}^d$ , for every fixed  $T > 0$ . Now notice that the functions  $\tilde{v}(t, x) := v(t, x) - g(x)$  and  $\tilde{w}(t, x) := w(t, x) - g(x)$  are, respectively, an upper semicontinuous subsolution and a lower semicontinuous supersolution of  $(\text{HJ}_1)$  with modified Hamiltonian  $\tilde{H}(x, p) = -\text{tr}(A(x)D_x^2 g) + H(x, p + D_x g)$ . Therefore, it suffices to establish the result for  $\tilde{v}$ ,  $\tilde{w}$  and  $\tilde{H}$  in place of  $v$ ,  $w$  and  $H$ , respectively. Notice that, for every fixed  $T > 0$ , the function  $\tilde{v}$  is bounded in  $[0, T] \times \mathbb{R}^d$ , while  $\tilde{w}$  satisfies

$$\tilde{w}(t, x) \geq -2C_T + \tilde{w}(0, x) \geq -2C_T + \tilde{v}(0, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d$$

for some constant  $C_T > 0$ , by definition of  $\mathcal{K}$ . The assertion now follows from Proposition 1.4 in [12] with  $U := \mathbb{R}^d$ .  $\square$

If  $A \equiv 0$  and  $H$  satisfies (H1)-(H2), then the Cauchy problem for  $(\text{HJ}_1)$  is always  $\mathcal{K}_*$ -well-posed, as stated in Theorem 2.5 below. But first we record a slightly more general comparison result than (b) for future use.

**Proposition 2.4.** *Let  $A \equiv 0$  and  $H$  satisfy hypotheses (H1)-(H2). Suppose that  $v \in \text{USC}([0, +\infty) \times \mathbb{R}^d)$  and  $w \in \text{LSC}([0, +\infty) \times \mathbb{R}^d)$  are, respectively, a sub and a supersolution of  $(\text{HJ}_1)$  belonging to  $\mathcal{K}_*$ . Then*

$$w(t, x) - v(t, x) \leq \sup_{\mathbb{R}^d} (w(0, \cdot) - v(0, \cdot)) \quad \text{for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$

A proof can be found in [15, Proposition A.2].

**Theorem 2.5.** *Let  $A \equiv 0$  and  $H$  satisfy hypotheses (H1)-(H2). Then the Cauchy problem for  $(\text{HJ}_1)$  is  $\mathcal{K}_*$ -well-posed. Moreover, for each  $g \in \text{UC}([0, +\infty) \times \mathbb{R}^d)$ , the solution  $u$  belongs to  $\text{UC}([0, +\infty) \times \mathbb{R}^d)$ . Furthermore, if  $g \in \text{Lip}(\mathbb{R}^d)$ , the solution  $u$  is Lipschitz continuous in  $[0, +\infty) \times \mathbb{R}^d$  and its Lipschitz constant depends only on  $\|Dg\|_{L^\infty(\mathbb{R}^d)}$  and the functions  $\alpha, \beta$ .*

*Proof.* Let us first assume  $g \in \text{Lip}(\mathbb{R}^d)$ . Take a function  $f \in C^1(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d)$  such that  $\|g - f\|_\infty < 1$  and a constant  $C$  satisfying

$$C > \max\{-\alpha(\|Dg\|_\infty), \beta(\|Dg\|_\infty)\}.$$

Notice that, in particular,  $C > \|H(x, Dg(x))\|_\infty$ . Choose  $n \in \mathbb{N}$  large enough so that the Hamiltonian

$$H_n(x, p) := \max\{H(x, p), 2\beta(|p|) - n\}, \quad (x, p) \in \mathbb{R}^d \times \mathbb{R}^d$$

satisfies

$$H_n = H \quad \text{on } U := \{(x, p) \in \mathbb{R}^d \times \mathbb{R}^d : H_n(x, p) < 3C\}. \quad (2.4)$$

The modified Hamiltonian  $H_n$  satisfies the additional condition (H1) in [10, p. 64], thus we can apply [10, Theorem 8.2] and infer the existence of a function  $\tilde{u} \in \text{Lip}([0, +\infty) \times \mathbb{R}^d)$  which solves  $(\text{HJ}_1)$  in  $[0, +\infty) \times \mathbb{R}^d$  with  $\tilde{H}(x, p) = H_n(x, p + D_x f)$  in place of  $H$  and initial condition  $\tilde{u}(0, \cdot) = g - f$  on  $\mathbb{R}^d$ . Moreover, it is not hard to see that the functions  $g(x) - f(x) - 2Ct$  and  $g(x) - f(x) + 2Ct$  are, respectively, a sub and a supersolution of

(HJ<sub>1</sub>) in  $[0, +\infty) \times \mathbb{R}^d$  with  $\tilde{H}$  in place of  $H$ . By arguing as in the proof of [10, Theorem 8.2] we obtain  $\|\partial_t \tilde{u}\|_\infty \leq 2C$ , in particular

$$H_n(x, D_x \tilde{u}(t, x) + Df(x)) \leq 2C \quad \text{for a.e. } (t, x) \in (0, +\infty) \times \mathbb{R}^d. \quad (2.5)$$

Then the function  $u := \tilde{u} + f$  belongs to  $\text{Lip}([0, +\infty) \times \mathbb{R}^d)$  and solves (HJ<sub>1</sub>) with  $H_n$  in place of  $H$  and initial condition  $u(0, \cdot) = g$  on  $\mathbb{R}^d$ . It is a standard fact that (2.5) implies that  $(x, D_x \phi(t, x)) \in U$  for any sub or supertangent  $\phi$  to  $u$  at  $(t, x) \in (0, +\infty) \times \mathbb{R}^d$ . In view of (2.4), we finally infer that  $u$  is a solution of (HJ<sub>1</sub>) in  $[0, +\infty) \times \mathbb{R}^d$  as well.

Let now assume that  $g \in \text{UC}(\mathbb{R}^d)$ . Then by Lemma 2.1 there exists a sequence of functions  $g_n \in \text{Lip}(\mathbb{R}^d)$  uniformly converging to  $g$  on  $\mathbb{R}^d$ . Let us denote by  $u_n$  the corresponding Lipschitz solution to (HJ<sub>1</sub>) with initial datum  $g_n$ . By Proposition 2.3 we have

$$\|u_m - u_n\|_{L^\infty([0, +\infty) \times \mathbb{R}^d)} \leq \|g_m - g_n\|_{L^\infty(\mathbb{R}^d)},$$

that is,  $(u_n)_n$  is a Cauchy sequence in  $[0, +\infty) \times \mathbb{R}^d$  with respect to the sup-norm. Hence the Lipschitz continuous functions  $u_n$  uniformly converge to a function  $u$  on  $[0, +\infty) \times \mathbb{R}^d$ , which is therefore uniformly continuous. By the stability of the notion of viscosity solution, we conclude that  $u$  is a solution of (HJ<sub>1</sub>) with initial datum  $g$ . The remainder of the assertion is a straightforward consequence of Proposition 2.4.  $\square$

In the case  $A \neq 0$ , we need to introduce additional assumptions on the Hamiltonian to be sure that the Cauchy problem for the viscous equation (HJ<sub>1</sub>) is  $\mathcal{K}_*$ -well-posed. Our examples in Section 4 will satisfy these conditions.

**Definition 2.6.** We shall say that a function  $H \in C(\mathbb{R}^d \times \mathbb{R}^d)$  belongs to the class  $\mathcal{H}(\gamma, \alpha_0, \beta_0)$  for some constants  $\alpha_0, \beta_0 > 0$  and  $\gamma > 1$  if it satisfies the following inequalities:

- (i)  $\alpha_0 |p|^\gamma - 1/\alpha_0 \leq H(x, p) \leq \beta_0(|p|^\gamma + 1)$  for all  $x, p \in \mathbb{R}^d$ ;
- (ii)  $|H(x, p) - H(y, p)| \leq \beta_0(|p|^\gamma + 1)|x - y|$  for all  $x, y, p \in \mathbb{R}^d$ ;
- (iii)  $|H(x, p) - H(x, q)| \leq \beta_0(|p| + |q| + 1)^{\gamma-1} |p - q|$  for all  $x, p, q \in \mathbb{R}^d$ .

Clearly, any  $H \in \mathcal{H}(\gamma, \alpha_0, \beta_0)$  satisfies (H1)-(H2). The following comparison principle holds:

**Proposition 2.7.** Assume  $A$  satisfies (A1)-(A2) and  $H \in \mathcal{H}(\gamma, \alpha_0, \beta_0)$ . Let  $v \in \text{USC}([0, +\infty) \times \mathbb{R}^d)$  and  $w \in \text{LSC}([0, +\infty) \times \mathbb{R}^d)$  be, respectively, a sub and a supersolution of (HJ<sub>1</sub>) belonging to  $\mathcal{K}$  and such that either  $v(0, \cdot)$  or  $w(0, \cdot)$  is in  $\text{UC}(\mathbb{R}^d)$ . Then

$$v(t, x) - w(t, x) \leq \sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)) \quad \text{for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$

The assertion follows by arguing as in the proof of Proposition 2.3 and by using [12, Theorem 3.1] in place of [12, Proposition 1.4].

From [12] we infer the following result.

**Theorem 2.8.** Let  $A$  satisfy (A1)-(A2) and  $H \in \mathcal{H}(\gamma, \alpha_0, \beta_0)$ . Then the Cauchy problem for (HJ<sub>1</sub>) is  $\mathcal{K}_*$ -well-posed. Moreover, for every  $g \in \text{UC}(\mathbb{R}^d)$ , the solution  $u$  belongs to  $\text{UC}([0, +\infty) \times \mathbb{R}^d)$ . If  $g \in \text{Lip}^2(\mathbb{R}^d)$ , then the solution  $u$  is Lipschitz continuous on  $[0, +\infty) \times \mathbb{R}^d$ .



*Proof.* Given  $g \in \text{UC}(\mathbb{R}^d)$ , we pick  $f \in C^\infty(\mathbb{R}^d) \cap \text{Lip}^3(\mathbb{R}^d)$  such that  $\|g - f\|_\infty < 1$ . According to Theorem 3.2 in [12], there exists  $\tilde{u} \in \text{UC}([0, +\infty) \times \mathbb{R}^d)$  that solves (HJ<sub>1</sub>) with modified Hamiltonian  $\tilde{H}(x, p) = -\text{tr}(A(x)D_x^2 f) + H(x, p + D_x f)$  and initial datum  $g - f$ . Moreover,  $\tilde{u}$  is bounded in cylinders of the form  $[0, T] \times \mathbb{R}^d$ , for every fixed  $T > 0$ . It is easily seen that  $u := \tilde{u} + f$  is in  $\text{UC}([0, +\infty) \times \mathbb{R}^d) \cap \mathcal{K} \subset \mathcal{K}_*$  and solves (HJ<sub>1</sub>) with initial condition  $u(0, \cdot) = g$  on  $\mathbb{R}^d$ .

If  $g \in \text{Lip}^2(\mathbb{R}^d)$ , then we can ensure that  $f$  also satisfies the inequalities

$$\|Df - Dg\|_\infty < 1, \quad \|D^2 f - D^2 g\|_\infty < 2\|D^2 g\|_\infty. \quad (2.6)$$

In view of Theorem 3.2 in [12], we conclude that  $\tilde{u}$ , and hence  $u$ , are Lipschitz continuous in  $[0, +\infty) \times \mathbb{R}^d$ . The remainder of the assertion is a straightforward consequence of Proposition 2.7.  $\square$

**Remark 2.9.** From Theorem 3.2 in [12] and in view of (2.6), we can also infer that the Lipschitz constant  $\kappa$  of the solution  $u$  with initial datum  $g \in \text{Lip}^2(\mathbb{R}^d)$  is a locally bounded function of  $\|Dg\|_{L^\infty(\mathbb{R}^d)}$  and  $\|D^2 g\|_{L^\infty(\mathbb{R}^d)}$ . More precisely,  $\kappa = \tilde{\kappa} \left( \|Dg\|_{L^\infty(\mathbb{R}^d)}, \|D^2 g\|_{L^\infty(\mathbb{R}^d)} \right)$  for some function  $\tilde{\kappa} : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$  which only depends on  $\Lambda_A$ ,  $\alpha_0$ ,  $\beta_0$  and  $\gamma > 1$  and is locally bounded in its arguments. This remark will be needed in Section 4.2.

### 3. HOMOGENIZATION: FROM LINEAR TO GENERAL INITIAL DATA

In this section we consider viscous and non-viscous Hamilton-Jacobi equations of the form (HJ<sub>ε</sub>) where the matrix  $A$  and the Hamiltonian  $H$  satisfy (A1)-(A2) and (H1)-(H2) respectively. We shall also assume that the Cauchy problem for (HJ<sub>1</sub>) is  $\mathcal{K}_*$ -well-posed, in the sense of Definition 2.2. Note that this is equivalent to the  $\mathcal{K}_*$ -well-posedness of the Cauchy problem for (HJ<sub>ε</sub>), for some (and, thus, for all)  $\varepsilon > 0$ . It suffices to remark that  $u \in \mathcal{K}_*$  is a continuous solution to (HJ<sub>1</sub>) if and only if the function  $u^\varepsilon(t, x) := \varepsilon u(t/\varepsilon, x/\varepsilon)$  is a continuous solution to (HJ<sub>ε</sub>) which belongs to  $\mathcal{K}_*$ .

We are interested in the asymptotic behavior of the solutions to (HJ<sub>ε</sub>) when  $\varepsilon \rightarrow 0^+$ . We would like to show that, in order to establish a homogenization result for (HJ<sub>ε</sub>), it is sufficient to consider only linear initial data. To this aim, for every fixed  $\theta \in \mathbb{R}^d$  and  $\varepsilon > 0$ , we denote by  $u_\theta$  and  $u_\theta^\varepsilon$  the unique continuous functions in  $\mathcal{K}_*$  that solve (HJ<sub>1</sub>) and (HJ<sub>ε</sub>) respectively subject to the initial condition  $u_\theta(0, x) = u_\theta^\varepsilon(0, x) = \langle \theta, x \rangle$ .

The main result of this section is the following theorem.

**Theorem 3.1.** *Let  $A$  and  $H$  satisfy hypotheses (A1)-(A2) and (H1)-(H2), respectively. Assume that the Cauchy problem for (HJ<sub>1</sub>) is  $\mathcal{K}_*$ -well-posed, and that either one of the following conditions is satisfied:*

(H3) *there exists a modulus of continuity  $m$  such that*

$$|H(x, p_1) - H(x, p_2)| \leq m(|p_1 - p_2|) \quad \text{for all } x \in \mathbb{R}^d \text{ and } p_1, p_2 \in \mathbb{R}^d;$$

(L) *for every  $\theta \in \mathbb{R}^d$ , there exists a constant  $\kappa = \kappa(\theta)$  such that*

$$|u_\theta(t, x) - u_\theta(t, y)| \leq \kappa|x - y| \quad \text{for all } x, y \in \mathbb{R}^d \text{ and } t \geq 0.$$

*Finally, suppose that there exists a continuous and coercive Hamiltonian  $\overline{H} : \mathbb{R}^d \rightarrow \mathbb{R}$  such that, for every  $\theta \in \mathbb{R}^d$*

$$u_\theta^\varepsilon(t, x) \rightrightarrows_{loc} \langle \theta, x \rangle - t\overline{H}(\theta) \quad \text{in } [0, +\infty) \times \mathbb{R}^d \text{ as } \varepsilon \rightarrow 0. \quad (3.1)$$

Then, for every  $g \in \text{UC}(\mathbb{R}^d)$ , the unique continuous function  $u^\varepsilon$  in  $\mathcal{K}_*$  solving  $(\text{HJ}_\varepsilon)$  with initial condition  $u^\varepsilon(0, \cdot) = g$  converges, locally uniformly in  $[0, +\infty) \times \mathbb{R}^d$  as  $\varepsilon \rightarrow 0^+$ , to the unique solution  $\bar{u} \in \text{UC}([0, +\infty) \times \mathbb{R}^d)$  of

$$\partial_t \bar{u} + \bar{H}(D_x \bar{u}) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d \quad (3.2)$$

with the initial condition  $\bar{u}(0, \cdot) = g$ .

**Remark 3.2.** It is easy to see that, by uniqueness,  $u_\theta^\varepsilon(t, x) = \varepsilon u_\theta(t/\varepsilon, x/\varepsilon)$ . Therefore, the hypothesis (L) amounts to requiring that the functions  $\{u_\theta^\varepsilon(t, \cdot) : 0 < \varepsilon \leq 1, t \geq 0\}$  are equi-Lipschitz in  $\mathbb{R}^d$ . A similar remark applies to condition (L') below.

To keep the proof of Theorem 3.1 concise, we shall first prove the following fact.

**Proposition 3.3.** *Let us assume that all the hypotheses of Theorem 3.1 are in force. Let  $g \in \text{UC}(\mathbb{R}^d)$  and, for every  $\varepsilon > 0$ , let us denote by  $u^\varepsilon$  the unique continuous function in  $\mathcal{K}_*$  that solves  $(\text{HJ}_\varepsilon)$  subject to the initial condition  $u^\varepsilon(0, \cdot) = g$ . Set*

$$\begin{aligned} u^*(t, x) &:= \lim_{r \rightarrow 0} \sup \{u^\varepsilon(s, y) : (s, y) \in (t-r, t+r) \times B_r(x), 0 < \varepsilon < r\}, \\ u_*(t, x) &:= \lim_{r \rightarrow 0} \inf \{u^\varepsilon(s, y) : (s, y) \in (t-r, t+r) \times B_r(x), 0 < \varepsilon < r\}. \end{aligned}$$

Let us assume that  $u^*$  and  $u_*$  are finite valued. Then

- (i)  $u^* \in \text{USC}([0, +\infty) \times \mathbb{R}^d)$  and it is a viscosity subsolution of (3.2);
- (ii)  $u_* \in \text{LSC}([0, +\infty) \times \mathbb{R}^d)$  and it is a viscosity supersolution of (3.2).

Theorem 3.1 follows readily from Proposition 3.3, as we show now.

*Proof of Theorem 3.1.* Let us first assume  $g \in C^2(\mathbb{R}^d) \cap \text{Lip}^2(\mathbb{R}^d)$ . Take a constant  $M$  large enough so that

$$M > \|\text{tr}(A(x)D_x^2 g(x))\|_\infty + \|H(x, Dg(x))\|_\infty.$$

Then the functions  $u_-(t, x) := g(x) - Mt$  and  $u_+(t, x) := g(x) + Mt$  are, respectively, a Lipschitz continuous sub and supersolution of  $(\text{HJ}_\varepsilon)$  for every  $0 < \varepsilon \leq 1$ . By Proposition 2.3, we get  $u_- \leq u^\varepsilon \leq u_+$  in  $[0, +\infty) \times \mathbb{R}^d$  for every  $0 < \varepsilon \leq 1$ . By the definition of relaxed semilimits we infer

$$u_-(t, x) \leq u_*(t, x) \leq u^*(t, x) \leq u_+(t, x) \quad \text{for all } (t, x) \in [0, +\infty) \times \mathbb{R}^d,$$

in particular,  $u_*$ ,  $u^*$  satisfy  $u_*(0, \cdot) = u^*(0, \cdot) = g$  on  $\mathbb{R}^d$  and belong to  $\mathcal{K}_*$ . By Proposition 3.3, we know that  $u^*$  and  $u_*$  are, respectively, an upper semicontinuous subsolution and a lower semicontinuous supersolution of the effective equation (3.2). We can therefore apply the Comparison Principle for (3.2) stated in Proposition 2.4 to obtain  $u^* \leq u_*$  on  $[0, +\infty) \times \mathbb{R}^d$ . Since the opposite inequality holds by the definition of upper and lower relaxed semilimits, we conclude that the function

$$\bar{u}(t, x) := u_*(t, x) = u^*(t, x) \quad \text{for all } (t, x) \in [0, +\infty) \times \mathbb{R}^d$$

is the unique continuous viscosity solution of (3.2) in  $\mathcal{K}_*$  such that  $\bar{u}(0, \cdot) = g$  on  $\mathbb{R}^d$ . Furthermore, by Theorem 2.5,  $\bar{u}$  is Lipschitz continuous on  $[0, +\infty) \times \mathbb{R}^d$ . The fact that the relaxed semilimits coincide implies that  $u^\varepsilon$  converge locally uniformly in  $[0, +\infty) \times \mathbb{R}^d$  to  $\bar{u}$ , see for instance [1, Lemma 6.2, p. 80].

When the initial datum  $g$  is just uniformly continuous on  $[0, +\infty) \times \mathbb{R}^d$ , the result easily follows from the above and Lemma 2.1 by approximating  $g$  with a sequence of initial

data belonging to  $C^2(\mathbb{R}^d) \cap \text{Lip}^2(\mathbb{R}^d)$  and by making use of the contraction property (b) of Definition 2.2 for  $(\text{HJ}_\varepsilon)$ .  $\square$

*Proof of Proposition 3.3.* The fact that  $u^*$  and  $u_*$  are upper and lower semicontinuous on  $[0, +\infty) \times \mathbb{R}^d$  is an immediate consequence of their definition. Let us prove (i), i.e. that  $u^*$  is a subsolution of (3.2). The proof of (ii) is analogous.

We make use of Evans's perturbed test function method, see [16]. Let us assume, by contradiction, that  $u^*$  is not a subsolution of (3.2). Then there exists a function  $\phi \in C^2((0, +\infty) \times \mathbb{R}^d)$  that is a strict supertangent of  $u^*$  at some point  $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^d$  and for which the subsolution test fails, i.e.

$$\partial_t \phi(t_0, x_0) + \overline{H}(D_x \phi(t_0, x_0)) > 3\delta \quad (3.3)$$

for some  $\delta > 0$ . For  $r > 0$  define  $V_r := (t_0 - r, t_0 + r) \times B_r(x_0)$ . Choose  $r_0 > 0$  to be small enough so that  $V_{r_0}$  is compactly contained in  $(0, +\infty) \times \mathbb{R}^d$  and  $u^* - \phi$  attains a strict local maximum at  $(t_0, x_0)$  in  $V_{r_0}$ . In particular, we have for every  $r \in (0, r_0)$

$$\max_{\partial V_r} (u^* - \phi) < \max_{\overline{V_r}} (u^* - \phi) = (u^* - \phi)(t_0, x_0). \quad (3.4)$$

Let us set  $\theta := D_x \phi(t_0, x_0)$  and for every  $\varepsilon > 0$  denote by  $u_\theta^\varepsilon$  the unique continuous function in  $\mathcal{K}_*$  that solves  $(\text{HJ}_\varepsilon)$  subject to the initial condition  $u_\theta^\varepsilon(0, x) = \langle \theta, x \rangle$ . We claim that there is an  $r \in (0, r_0)$  such that the function

$$\phi^\varepsilon(t, x) := \phi(t, x) + u_\theta^\varepsilon(t, x) - (\langle \theta, x \rangle - t\overline{H}(\theta))$$

is a supersolution of  $(\text{HJ}_\varepsilon)$  in  $V_r$  for every  $\varepsilon > 0$  small enough. Indeed, by a direct computation we first get

$$\begin{aligned} \partial_t \phi^\varepsilon - \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 \phi^\varepsilon \right) + H \left( \frac{x}{\varepsilon}, D_x \phi^\varepsilon \right) &= \partial_t \phi + \overline{H}(\theta) - \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 \phi \right) \\ + \partial_t u_\theta^\varepsilon - \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 u_\theta^\varepsilon \right) + H \left( \frac{x}{\varepsilon}, D_x u_\theta^\varepsilon + D_x \phi - \theta \right) & \end{aligned} \quad (3.5)$$

in the viscosity sense in  $V_r$ . Using (3.3), assumption (A1), and the fact that  $\phi$  is of class  $C^2$ , we get that there is an  $r \in (0, r_0)$  such that for all sufficiently small  $\varepsilon > 0$  and all  $(t, x) \in V_r$

$$\partial_t \phi(t, x) + \overline{H}(\theta) - \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 \phi \right) > 3\delta - \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 \phi \right) > 2\delta.$$

Moreover, by taking into account either (H3) or (L) (together with Remark 3.2) and (H1), we can further reduce  $r$  if necessary to get

$$H \left( \frac{x}{\varepsilon}, Du_\theta^\varepsilon + D_x \phi - \theta \right) > H \left( \frac{x}{\varepsilon}, Du_\theta^\varepsilon \right) - \delta \quad \text{in the viscosity sense in } V_r.$$

Plugging these relations into (3.5) and using the fact that  $u_\theta^\varepsilon$  is a solution of  $(\text{HJ}_\varepsilon)$ , we finally get

$$\begin{aligned} \partial_t \phi^\varepsilon - \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 \phi^\varepsilon \right) + H \left( \frac{x}{\varepsilon}, D_x \phi^\varepsilon \right) \\ > \delta + \partial_t u_\theta^\varepsilon - \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 u_\theta^\varepsilon \right) + H \left( \frac{x}{\varepsilon}, Du_\theta^\varepsilon \right) = \delta > 0 \end{aligned}$$

in the viscosity sense in  $V_r$ , thus showing that  $\phi^\varepsilon$  is a supersolution of  $(\text{HJ}_\varepsilon)$  in  $V_r$ . We now need a comparison principle for equation  $(\text{HJ}_\varepsilon)$  in  $V_r$  applied to  $\phi^\varepsilon$  and  $u^\varepsilon$  to infer that

$$\sup_{V_r} (u^\varepsilon - \phi^\varepsilon) \leq \max_{\partial V_r} (u^\varepsilon - \phi^\varepsilon).$$

If condition (H3) holds, we can apply [11, Theorem 3.3 and Section 5.C]. If (L) is satisfied, then  $D_x \phi^\varepsilon \in L^\infty(V_r)$  and a standard argument (see, for instance, Proposition 1.4 in [12]) shows that the comparison principle for  $(\text{HJ}_\varepsilon)$  in  $V_r$  holds in this case as well. Now notice that, by the assumption (3.1),  $\phi^\varepsilon \rightrightarrows_{\text{loc}} \phi$  in  $\overline{V}_r$ . Taking the limsup of the last inequality as  $\varepsilon \rightarrow 0^+$  we obtain

$$\sup_{V_r}(u^* - \phi) \leq \overline{\lim}_{\varepsilon \rightarrow 0^+} \sup_{V_r}(u^\varepsilon - \phi^\varepsilon) \leq \overline{\lim}_{\varepsilon \rightarrow 0^+} \max_{\partial V_r}(u^\varepsilon - \phi^\varepsilon) \leq \max_{\partial V_r}(u^* - \phi),$$

a contradiction with (3.4). This proves that  $u^*$  is a subsolution of (3.2).  $\square$

Notice that in Theorem 3.1 and Proposition 3.3 we have assumed *as a hypothesis* that  $\overline{H} : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuous and coercive. While the latter property is inherited from that of  $H$ , as we show below, the continuity is not guaranteed a priori, but is deduced from the way the effective Hamiltonian is obtained. Our next result shows that this property can be, for instance, deduced when the bounds on the derivatives of the functions  $u_\theta^\varepsilon$  with respect to  $x$  are locally uniform with respect to  $\theta$ .

**Proposition 3.4.** *Let  $A$  and  $H$  satisfy hypotheses (A1)-(A2) and (H1)-(H2), respectively, and the Cauchy problem for  $(\text{HJ}_1)$  be  $\mathcal{K}_*$ -well-posed. Assume that, for every  $\theta \in \mathbb{R}^d$ , the convergence stated in (3.1) holds for some function  $\overline{H} : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then  $\overline{H}$  satisfies assumption (H2). Let us furthermore assume either condition (H3) or the following:*

(L') *for every  $r > 0$ , there exists a constant  $\kappa_r$  such that*

$$|u_\theta(t, x) - u_\theta(t, y)| \leq \kappa_r |x - y| \quad \text{for all } x, y \in \mathbb{R}^d, t \geq 0 \text{ and } \theta \in B_r.$$

*Then, for every  $r > 0$ , there exists a continuity modulus  $m_r$  such that*

$$|u_{\theta_1}^\varepsilon(t, x) - u_{\theta_2}^\varepsilon(t, x)| \leq t m_r (|\theta_1 - \theta_2|) + |x| |\theta_1 - \theta_2| \quad (3.6)$$

*for every  $x \in \mathbb{R}^d$  and  $\theta_1, \theta_2 \in B_r$ . In particular, the effective Hamiltonian  $\overline{H}$  is continuous.*

*Proof.* According to (3.1), the effective Hamiltonian is defined by the following formula:

$$\overline{H}(\theta) = \lim_{\varepsilon \rightarrow 0^+} -u_\theta^\varepsilon(1, 0) \quad \text{for every } \theta \in \mathbb{R}^d. \quad (3.7)$$

Let us first show that  $\overline{H}$  satisfies (H2). Let us fix  $\theta \in \mathbb{R}^d$ . It is easily seen that the functions  $u_-(t, x) = \langle \theta, x \rangle - t\beta(|\theta|)$  and  $u_+(t, x) = \langle \theta, x \rangle - t\alpha(|\theta|)$  are classical sub and supersolutions to  $(\text{HJ}_\varepsilon)$  for every  $\varepsilon > 0$ , respectively. By Proposition 2.3 we get  $u_- \leq u_\theta^\varepsilon \leq u_+$  in  $[0, +\infty) \times \mathbb{R}^d$ , and the assertion immediately follows from this in view of (3.7).

To prove (3.6), we let  $v_\theta^\varepsilon(t, x) := u_\theta^\varepsilon(t, x) - \langle \theta, x \rangle$  for every  $\theta \in \mathbb{R}^d$ . Then  $v_\theta^\varepsilon$  is a solution of

$$\partial_t v_\theta^\varepsilon - \varepsilon \operatorname{tr} \left( A \left( \frac{x}{\varepsilon} \right) D_x^2 v_\theta^\varepsilon \right) + H \left( \frac{x}{\varepsilon}, \theta + D_x v_\theta^\varepsilon \right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d \quad (3.8)$$

satisfying  $v_\theta^\varepsilon(0, x) = 0$  in  $\mathbb{R}^d$ . If hypothesis (L') is in force, then, in view of Remark 3.2, for every  $r > 0$  there exists  $\rho(r) > 0$  such that  $\|D_x v_\theta^\varepsilon\|_\infty < \rho(r)$  for every  $0 < \varepsilon \leq 1$  and  $\theta \in B_r$ . Let us fix  $r > 0$  and denote by  $m_r$  a modulus of continuity such that

$$|H(x, p_1) - H(x, p_2)| \leq m_r(|p_1 - p_2|) \quad \text{for all } x \in \mathbb{R}^d \text{ and } p_1, p_2 \in B_{r+\rho(r)}.$$

If, on the other hand, hypothesis (H3) is in force, then the above inequality holds with  $m_r$  independent of  $r$ . Take  $\theta_1, \theta_2 \in B_r$ . Then for every  $\varepsilon > 0$

$$\left| H\left(\frac{x}{\varepsilon}, \theta_1 + D_x v_{\theta_1}^\varepsilon\right) - H\left(\frac{x}{\varepsilon}, \theta_2 + D_x v_{\theta_1}^\varepsilon\right) \right| \leq m_r (|\theta_1 - \theta_2|)$$

in the viscosity sense in  $(0, +\infty) \times \mathbb{R}^d$ . We infer that the functions  $v_{\theta_1}^\varepsilon(t, x) - t m_r (|\theta_1 - \theta_2|)$  and  $v_{\theta_1}^\varepsilon(t, x) + t m_r (|\theta_1 - \theta_2|)$  are, respectively, a sub and a supersolution of (3.8) with  $\theta := \theta_2$ . By Proposition 2.3 we conclude that

$$|v_{\theta_1}^\varepsilon(t, x) - v_{\theta_2}^\varepsilon(t, x)| \leq t m_r (|\theta_1 - \theta_2|).$$

By the definition of  $v_\theta^\varepsilon$ , we get (3.6), and, in view of (3.7), we obtain, in particular,

$$|\overline{H}(\theta_1) - \overline{H}(\theta_2)| \leq m_r (|\theta_1 - \theta_2|) \quad \text{for all } \theta_1, \theta_2 \in B_r,$$

yielding the asserted continuity of  $\overline{H}$ .  $\square$

#### 4. HOMOGENIZATION IN THE STATIONARY ERGODIC SETTING

**4.1. Stationary ergodic framework.** In this section we recall basic definitions and discuss some of the implications of stationarity and ergodicity for the results of Section 3.

We denote by  $(\Omega, \mathcal{F}, \mathbb{P})$  a *probability space*, where  $\Omega$  is a sample space,  $\mathcal{F}$  is a sigma-algebra of subsets of  $\Omega$  and  $\mathbb{P}$  is a probability measure on  $(\Omega, \mathcal{F})$ . We shall denote by  $\mathcal{B}(\mathbb{R}^d)$  the  $\sigma$ -algebra of Borel subsets of  $\mathbb{R}^d$ . When defining measurability we shall always work with measurable spaces  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ ,  $(\Omega, \mathcal{F})$ , and the product space  $\mathbb{R}^d \times \Omega$  endowed with the corresponding product  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$ . Polish spaces  $C(\mathbb{R}^d)$  and  $C(\mathbb{R}^d \times \mathbb{R}^d)$  are considered with the topology of locally uniform convergence on  $\mathbb{R}^d$  and  $\mathbb{R}^d \times \mathbb{R}^d$  respectively and are equipped with their Borel  $\sigma$ -algebras.

A *d-dimensional dynamical system of shifts*  $(\tau_x)_{x \in \mathbb{R}^d}$  is defined as a family of mappings  $\tau_x : \Omega \rightarrow \Omega$  which satisfy the following properties:

- (1) (*group property*)  $\tau_0 = id$ ,  $\tau_{x+y} = \tau_x \circ \tau_y$ ;
- (2) (*preservation of measure*)  $\tau_x : \Omega \rightarrow \Omega$  is measurable and  $\mathbb{P}(\tau_x E) = \mathbb{P}(E)$  for every  $E \in \mathcal{F}$ ;
- (3) (*joint measurability*) the map  $(x, \omega) \mapsto \tau_x \omega$  from  $\mathbb{R}^d \times \Omega$  to  $\Omega$  is measurable.

The above properties guarantee (see [21, (7.2)]) that  $(\tau_x)_{x \in \mathbb{R}^d}$  is continuous, i.e.  $\lim_{|x| \rightarrow 0} \|f(\tau_x \omega) - f(\omega)\|_{L^2(\Omega)} = 0$  for every  $f \in L^2(\Omega)$ .

We make the crucial assumption that  $(\tau_x)_{x \in \mathbb{R}^d}$  is *ergodic*, i.e. any measurable function  $f : \Omega \rightarrow \mathbb{R}$  enjoying  $f(\tau_x \omega) = f(\omega)$  a.s. in  $\Omega$ , for any fixed  $x \in \mathbb{R}^d$ , is almost surely constant.

We shall say that a (measurable) random field  $H : \Omega \rightarrow C(\mathbb{R}^d \times \mathbb{R}^d)$  is *stationary with respect to the shifts*  $(\tau_x)_{x \in \mathbb{R}^d}$  if it admits the following representation:

$$H(x, p, \omega) = \tilde{H}(p, \tau_x \omega) \quad \text{for all } x, p \in \mathbb{R}^d \text{ and } \omega \in \Omega. \quad (4.1)$$

for some (measurable)  $\tilde{H} : \Omega \rightarrow C(\mathbb{R}^d)$ . Note that (4.1) and the group property (1) above immediately imply that

$$(S) \quad H(x + y, p, \omega) = H(x, p, \tau_y \omega) \quad \text{for all } x, y, p \in \mathbb{R}^d \text{ and } \omega \in \Omega.$$

Since random variables  $H(x, p, \cdot)$  and  $H(x+y, p, \cdot)$  have the same distribution due to (2), we can say informally that (S) expresses the desired feature of the underlying random medium: at different points in space the medium statistically “looks” the same. We also remark that every  $H$  satisfying (S) admits a representation of the form (4.1) (see, for instance, [13, Proposition 3.1]). Stationarity of a random process  $b : \Omega \rightarrow \mathbb{C}(\mathbb{R}^d)$  is defined in the same way simply by omitting  $p$  in (4.1).

We are interested in homogenization for solutions  $u^\varepsilon(t, x, \omega)$  of the Cauchy problem for equation  $(\text{HJ}_\varepsilon)$  with Hamiltonian  $H(x, p, \omega)$  on a set of  $\omega$  of full measure. We shall assume that all constants in the assumptions on  $A$  and  $H$ , i.e.  $\Lambda_A$ ,  $\alpha(\cdot)$ ,  $\beta(\cdot)$ , and the moduli of continuity of  $H$  on  $\mathbb{R}^d \times B_R$  for each  $R > 0$ , are independent of  $\omega$ .

We note that one of the consequences of is that the locally uniform convergence (3.1) of Theorem 3.1 follows from the a.s. convergence at  $t = 1$  (equivalently, at an arbitrary  $t > 0$ ) and  $x = 0$  as long as the family  $\{u_\theta^\varepsilon(1, \cdot, \omega), 0 < \varepsilon \leq 1, \omega \in \Omega\}$  is equi-continuous.

**Lemma 4.1.** *Let  $H$  and (all entries of)  $A$  be stationary in the sense of the above definition and satisfy hypotheses (H1)-(H2) and (A1)-(A2), respectively. We shall suppose that all uniform bounds are independent of  $\omega$ . Assume that, for every fixed  $\omega$ , the Cauchy problem for  $(\text{HJ}_1)$  is  $\mathcal{K}_*$ -well-posed, and that, for each  $\theta \in \mathbb{R}^d$ , there is a modulus of continuity  $m_\theta$  such that for all  $\varepsilon \in (0, 1]$  and  $\omega \in \Omega$*

$$|u_\theta^\varepsilon(1, x, \omega) - u_\theta^\varepsilon(1, 0, \omega)| \leq m_\theta(|x|) \quad \text{for all } x \in \mathbb{R}^d. \quad (4.2)$$

If, for each  $\theta \in \mathbb{R}^d$ , we have

$$\lim_{\varepsilon \rightarrow 0+} u_\theta^\varepsilon(1, 0, \omega) = -\overline{H}(\theta) \quad (4.3)$$

with probability 1, then the above convergence is locally uniform, i.e. with probability 1

$$u_\theta^\varepsilon(t, x, \omega) \rightrightarrows_{loc} \langle \theta, x \rangle - t\overline{H}(\theta) \quad \text{in } [0, +\infty) \times \mathbb{R}^d. \quad (4.4)$$

In particular, if the condition  $(L')$  from Proposition 3.4 is satisfied with  $\kappa_r$  independent of  $\omega$ , then there is a set  $\hat{\Omega} \subseteq \Omega$  of full measure such that the last convergence takes place for all  $\theta \in \mathbb{R}^d$  and all  $\omega \in \hat{\Omega}$ , and, thus, the conclusion of Theorem 3.1 holds for all  $\omega \in \hat{\Omega}$ .

The proof below is based on a by-now standard argument which appeared in, for instance, [22, pp. 1501-1502], [14, pp. 403-404], [4, Lemma 4.10]. It was later “distilled” into an abstract lemma in [6, Lemma 2.4], which is convenient for time-independent applications. Since  $u_\theta^\varepsilon$  is time dependent, we shall need an additional easy step.

*Proof.* Fix  $\theta \in \mathbb{R}^d$  and set

$$w(t, x, \omega) := u_\theta(t, x, \omega) - \langle \theta, x \rangle + t\overline{H}(\theta).$$

For any fixed  $\omega \in \Omega$ , the function  $w(\cdot, \cdot, \omega)$  is in  $\mathcal{K}_*$  and solves  $(\text{HJ}_1)$  with Hamiltonian  $H(\cdot, \theta + \cdot, \omega) - \overline{H}(\theta)$  and zero initial datum. By stationarity of the Hamiltonian and uniqueness of the solution, we conclude that  $w(t, \cdot, \cdot)$  is stationary in  $x$ , for every fixed  $t \geq 0$ . We claim that there exists a set  $\Omega_\theta$  of full measure such that, for every  $\omega \in \Omega_\theta$ ,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{y \in B_R} |u_\theta^\varepsilon(1, y, \omega) - \langle \theta, y \rangle + \overline{H}(\theta)| = 0 \quad \text{for all } R > 0. \quad (4.5)$$



To prove this, it suffices to apply [6, Lemma 2.4] with  $X_\varepsilon(x, \omega) := \varepsilon |w(1/\varepsilon, x, \omega)|$ . Indeed, by the rescaling  $u_\theta^\varepsilon(t, x, \omega) = \varepsilon u_\theta(t/\varepsilon, x/\varepsilon, \omega)$ , claim (4.5) is equivalent to

$$\mathbb{P}\left(\omega \in \Omega : \forall R > 0 \quad \limsup_{\varepsilon \rightarrow 0} \sup_{y \in B_{R/\varepsilon}} |X_\varepsilon(\cdot, \omega)| = 0\right) = 1.$$

Let us then check that such  $X_\varepsilon$  satisfy the conditions stated in the quoted lemma. The stationarity of  $X_\varepsilon$  follows from the stationarity of  $v$ . The a.s. convergence  $\lim_{\varepsilon \rightarrow 0} X_\varepsilon(0, \cdot) = 0$  is just a restatement of (4.3). The property

$$\mathbb{P}\left(\omega \in \Omega : \forall z \in \mathbb{R}^d \quad \lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \operatorname{osc}_{B_{r/\varepsilon}(z/\varepsilon)} X_\varepsilon(\cdot, \omega) = 0\right) = 1$$

is an immediate consequence of the assumption (4.2). Indeed,

$$\begin{aligned} \operatorname{osc}_{y \in B_{r/\varepsilon}(z/\varepsilon)} X_\varepsilon(y, \omega) &\leq \varepsilon \sup_{x, y \in B_{r/\varepsilon}(z/\varepsilon)} |w(1/\varepsilon, 0, \tau_x \omega) - w(1/\varepsilon, y - x, \tau_x \omega)| \\ &\leq 2r|\theta| + \sup_{|y-x| \leq 2r} |u_\theta^\varepsilon(1, 0, \tau_{x/\varepsilon} \omega) - u_\theta^\varepsilon(1, y - x, \tau_{x/\varepsilon} \omega)| \leq 2r|\theta| + m_\theta(2r). \end{aligned}$$

Let us proceed to show that, for every fixed  $\omega \in \Omega_\theta$ , the convergence (4.4) holds. We first take note of the following scaling relations:

$$u_\theta^\varepsilon(t, x, \omega) = t(\varepsilon/t) u_\theta(t/\varepsilon, x/\varepsilon, \omega) = t u_\theta^{\varepsilon/t}(1, x/t, \omega) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

Fix  $T > 0$ . Then for  $r \in (0, T)$  we obtain

$$\begin{aligned} &\sup_{r \leq t \leq T} \sup_{y \in B_R} |u_\theta^\varepsilon(t, y, \omega) - \langle \theta, y \rangle + t \overline{H}(\theta)| \\ &= \sup_{r \leq t \leq T} \sup_{y \in B_R} |t u_\theta^{\varepsilon/t}(1, y/t, \omega) - t \langle \theta, y/t \rangle + t \overline{H}(\theta)| \\ &\leq T \sup_{\varepsilon/T \leq \eta \leq \varepsilon/r} \sup_{z \in B_{R/r}} |u_\theta^\eta(1, z, \omega) - \langle \theta, z \rangle + \overline{H}(\theta)|. \end{aligned}$$

By (4.5), the right-hand side goes to 0 as  $\varepsilon \rightarrow 0^+$ . Finally, we use the uniform in  $\varepsilon$  (and  $\omega$ ) continuity of  $u^\varepsilon(t, 0, \omega)$  at  $t = 0$  implied by the condition  $\mathcal{K}_*$  and get that for all  $r \in (0, T)$

$$\begin{aligned} &\sup_{0 \leq t \leq r} \sup_{y \in B_R} |u_\theta^\varepsilon(t, y, \omega) - \langle \theta, y \rangle + t \overline{H}(\theta)| \\ &\leq \sup_{0 \leq t \leq r} \sup_{y \in B_R} |\varepsilon u_\theta(t/\varepsilon, y/\varepsilon, \omega) - \varepsilon u_\theta(0, y/\varepsilon, \omega)| + r |\overline{H}(\theta)| \leq \varepsilon a + r M_a + r |\overline{H}(\theta)|. \end{aligned}$$

The last expression goes to zero when we let  $\varepsilon \rightarrow 0^+$  and then  $r \rightarrow 0^+$ . This proves (4.4) for all  $\omega \in \Omega_\theta$ . The remainder of the statement with  $\hat{\Omega} = \cap_{\theta \in \mathbb{Q}^d} \Omega_\theta$  follows from (4.4), (L'), and the bound (3.6) in Proposition 3.4.  $\square$

**4.2. Homogenization for non-convex Hamiltonians.** The aim of the present section is to establish a homogenization result in the stationary ergodic setting for equations of the form  $(\text{HJ}_\varepsilon)$  in one space dimension, where the stationary random field  $H : \Omega \rightarrow \mathbb{C}(\mathbb{R} \times \mathbb{R})$  takes values in a special class of non-convex Hamiltonians, see Theorem 4.10 for details. The proof of this result is derived from a more general principle that we shall describe and prove first.

Let  $H_+, H_- : \Omega \rightarrow \mathbb{C}(\mathbb{R} \times \mathbb{R})$  be stationary random fields such that  $H_\pm(\cdot, \cdot, \omega) \in \mathcal{H}(\gamma, \alpha_0, \beta_0)$  for every  $\omega$  (see Definition 2.6), with  $\gamma > 1$ ,  $\alpha_0, \beta_0 > 0$  independent of  $\omega$ . In addition, we assume that

$$H_\pm(x, 0, \omega) = h_0 \quad \text{and} \quad H_+(x, p, \omega)p \leq H_-(x, p, \omega)p \quad \text{for all } (x, p, \omega) \in \mathbb{R}^2 \times \Omega, \quad (4.6)$$

for some constant  $h_0 \in \mathbb{R}$ . Let

$$H(x, p, \omega) = \min\{H_+(x, p, \omega), H_-(x, p, \omega)\} \quad \text{for all } (x, p, \omega) \in \mathbb{R}^2 \times \Omega. \quad (4.7)$$

Then  $H(\cdot, 0, \cdot) \equiv h_0$  on  $\mathbb{R} \times \Omega$  and, in view of (4.6), we also have

$$H(x, p, \omega) = H_+(x, p, \omega) \quad \text{if } p \geq 0 \quad \text{and} \quad H(x, p, \omega) = H_-(x, p, \omega) \quad \text{if } p \leq 0.$$

Note that  $H$  is stationary and, for every fixed  $\omega$ , belongs to the same class  $\mathcal{H}(\gamma, \alpha_0, \beta_0)$  as  $H_\pm$ . We let  $A(x, \omega)$  be a stationary process which satisfies (A1)-(A2) with  $\Lambda_A$  independent of  $\omega$ . We consider the family  $u^\varepsilon(\cdot, \cdot, \omega)$ ,  $\varepsilon \in (0, 1]$ , of solutions to the equation

$$u_t^\varepsilon - \varepsilon A\left(\frac{x}{\varepsilon}\right) u_{xx}^\varepsilon + H\left(\frac{x}{\varepsilon}, u_x^\varepsilon, \omega\right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}, \quad (\text{HJ}_\varepsilon^\omega)$$

subject to the initial condition  $u^\varepsilon(0, \cdot, \omega) = g \in \text{UC}(\mathbb{R})$ . These Cauchy problems are  $\mathcal{K}_*$ -well-posed thanks to Theorem 2.8.

We aim to show that equation  $(\text{HJ}_\varepsilon^\omega)$  homogenizes whenever this holds true with  $H_\pm$  in place of  $H$ . More precisely, we shall prove the following result.

**Theorem 4.2.** *Let  $H$  be given by (4.7), where  $H_\pm : \Omega \rightarrow \mathbb{C}(\mathbb{R} \times \mathbb{R})$  are stationary random fields satisfying (4.6) and such that  $H_\pm(\cdot, \cdot, \omega) \in \mathcal{H}(\gamma, \alpha_0, \beta_0)$  for every  $\omega$ , with constants  $\gamma > 1$ ,  $\alpha_0, \beta_0 > 0$  independent of  $\omega$ . Let us furthermore assume that there exist sets  $\Omega_\pm$  of full measure in  $\Omega$  such that  $(\text{HJ}_\varepsilon^\omega)$  with  $H_\pm$  in place of  $H$  homogenizes for all linear initial data  $g(x) = \theta x$ ,  $\theta \in \mathbb{R}$  and for every  $\omega \in \Omega_\pm$ , respectively. Let us denote by  $\overline{H}_\pm$  the associated (continuous and coercive) effective Hamiltonians. Then there exists a continuous and coercive Hamiltonian  $\overline{H} : \mathbb{R} \rightarrow \mathbb{R}$  such that, for every  $\omega \in \Omega_- \cap \Omega_+$  and every initial datum  $g \in \text{UC}(\mathbb{R})$ , the unique solutions  $u^\varepsilon(\cdot, \cdot, \omega) \in \text{UC}([0, +\infty) \times \mathbb{R}^d)$  of  $(\text{HJ}_\varepsilon^\omega)$  with the initial condition  $u^\varepsilon(0, \cdot, \omega) = g$  converge, locally uniformly in  $[0, +\infty) \times \mathbb{R}$  as  $\varepsilon \rightarrow 0^+$ , to the unique solution  $\overline{u} \in \text{UC}([0, +\infty) \times \mathbb{R}^d)$  of*

$$\begin{cases} \overline{u}_t + \overline{H}(\overline{u}_x) = 0 & \text{in } [0, +\infty) \times \mathbb{R} \\ \overline{u}(0, \cdot, \omega) = g & \text{in } \mathbb{R}. \end{cases}$$

Moreover,

$$\overline{H}(\theta) = \min\{\overline{H}_-(\theta), \overline{H}_+(\theta)\} \quad \text{for every } \theta \in \mathbb{R}.$$

Before dealing with the proof of Theorem 4.2, we derive some simple consequences.

**Corollary 4.3.** *Let  $H$  be as in the statement of Theorem 4.2 and let us additionally assume that  $H_\pm$  are convex in  $p$ , or level-set convex in  $p$  if  $A \equiv 0$ . Then there exists a set  $\hat{\Omega}$  of full measure in  $\Omega$  such that, for every  $\omega \in \hat{\Omega}$ , equation  $(\text{HJ}_\varepsilon^\omega)$  homogenizes for all  $g \in \text{UC}(\mathbb{R})$ .*

*Proof.* The assertion immediately follows from Theorem 4.2 and homogenization results for viscous and non-viscous Hamilton-Jacobi equations with stationary ergodic convex (or level set convex if  $A \equiv 0$ ) Hamiltonians [6, 13, 22, 27–29].  $\square$

**Remark 4.4.** Note that the effective Hamiltonian  $\overline{H}$  associated to  $(HJ_\varepsilon^\omega)$  is, in general, neither convex nor even level-set convex. Indeed, let  $b(\cdot, \omega) \in \text{Lip}(\mathbb{R})$  with a Lipschitz constant independent of  $\omega$ ,  $a \leq b(x, \omega) \leq 1/a$  in  $\mathbb{R} \times \Omega$  for some constant  $a \in (0, 1)$ . Let  $H_\pm(x, p, \omega) = |p|^2/2 \mp b(x, \omega)p$  and define

$$H(x, p, \omega) = \min\{H_+(x, p, \omega), H_-(x, p, \omega)\} = \frac{|p|^2}{2} - b(x, \omega)|p| \quad \text{in } \mathbb{R}^2 \times \Omega.$$

Then  $H, H_\pm \in \mathcal{H}(2, \alpha_0, \beta_0)$  for some  $\alpha_0, \beta_0 > 0$ . Set  $\hat{H}(p) := |p|^2/2 - a|p|$ . Let us denote by  $u_\theta^\varepsilon$  and  $\hat{u}_\theta^\varepsilon$  the solutions to  $(HJ_\varepsilon^\omega)$  with  $A = 1$  and Hamiltonians  $H$  and  $\hat{H}$  respectively, and the initial condition  $\theta x$ . Since  $H \leq \hat{H}$ , by the Comparison Principle stated in Proposition 2.7 we infer that  $\hat{u}_\theta^\varepsilon \leq u_\theta^\varepsilon$ . Therefore,

$$\overline{H}(\theta) = \lim_{\varepsilon \rightarrow 0} -u^\varepsilon(1, 0, \omega) \leq \lim_{\varepsilon \rightarrow 0} -\hat{u}^\varepsilon(1, 0) = \hat{H}(\theta) \quad \text{a.s. in } \omega,$$

in particular  $\overline{H}(\pm a) \leq \hat{H}(\pm a) < 0$ . Since  $\overline{H}(0) = 0$  and  $\overline{H}$  is coercive, the assertion follows.

We now proceed to prove Theorem 4.2. The idea we are going to exploit is that, if the initial datum  $g$  is monotone, the solution of the corresponding Cauchy problem associated to  $(HJ_\varepsilon^\omega)$  enjoys the same kind of monotonicity as  $g$  with respect to the  $x$  variable. Due to the scaling relation  $u_\theta^\varepsilon(t, x) = \varepsilon u_\theta(t/\varepsilon, x/\varepsilon, \omega)$ , it suffices to consider the case  $\varepsilon = 1$ .

**Proposition 4.5.** *Let  $A$  satisfy (A1)-(A2) and  $H \in \mathcal{H}(\gamma, \alpha_0, \beta_0)$  be such that  $H(\cdot, 0) \equiv h_0$  on  $\mathbb{R}$ , for some constant  $h_0 \in \mathbb{R}$ . Denote by  $w$  the unique continuous solution in  $\mathcal{K}_*$  of the Cauchy problem*

$$\begin{cases} w_t - A(x)w_{xx} + H(x, w_x) = 0 & \text{in } (0, +\infty) \times \mathbb{R}, \\ w(0, x) = g(x) & \text{in } \mathbb{R}, \end{cases} \quad (4.8)$$

where  $g \in \text{Lip}^2(\mathbb{R}^d)$ . Then

- (i) if  $g' \geq 0$ , then  $w(t, \cdot)$  is nondecreasing, for every fixed  $t \geq 0$ ;
- (ii) if  $g' \leq 0$ , then  $w(t, \cdot)$  is nonincreasing, for every fixed  $t \geq 0$ .

*Proof.* The existence and uniqueness of continuous  $w$  in  $\mathcal{K}_*$  follow from Theorem 2.8. By the same theorem,  $w$  is Lipschitz continuous in  $[0, +\infty) \times \mathbb{R}^d$ .

Let us now show monotonicity. Up to replacing  $H$  with  $H - h_0$  and  $w$  with  $w + h_0 t$ , we can assume without loss of generality that  $H(\cdot, 0) \equiv 0$  on  $\mathbb{R}$ . Suppose, for definiteness, that  $g' \geq 0$ . For technical reasons we need to perform a regularization of  $g$ ,  $A$  and of the nonlinearity  $H$ . To this aim, take a standard sequence of mollifiers  $(\rho_n)_n$  on  $\mathbb{R}$  and set  $g_n(x) := g * \rho_n(x)$ ,  $A_n(x) = A * \rho_n(x) + 1/n$  and  $H_n(x, p) := \hat{H}_n(x, p) - \hat{H}_n(x, 0)$  with

$$\hat{H}_n(x, p) := \int_{\mathbb{R} \times \mathbb{R}} \rho_n(y) \rho_n(q) H(x - y, p - q) dy dq, \quad (x, p) \in \mathbb{R} \times \mathbb{R}.$$

An easy check shows that  $A_n$  satisfies (A1)-(A2) with  $\Lambda_{A_n} = \Lambda_A + 1$  and that  $H_n$  belongs to the class  $\mathcal{H}(\gamma, \alpha_0, \beta_0)$  with constants  $\alpha_0, \beta_0$  independent of  $n$ , possibly different from the ones assumed for  $H$  at the beginning. Thus, the conditions of Theorem 2.8 are satisfied. Moreover, for every  $R > 0$ , there exists a constant  $C(R, n) > 0$  such that

$$|\partial_x A_n|, |\partial_p H_n|, |\partial_p^2 H_n|, |\partial_{xp}^2 H_n| \leq C(R, n) \quad \text{for all } x \in \mathbb{R} \text{ and } p \in B_R. \quad (4.9)$$

Let us denote by  $w_n$  the unique continuous solution of class  $\mathcal{K}_*$  of the problem (4.8) with  $H_n$  in place of  $H$ . Since  $\sup_n \|Dg_n\|_\infty + \|D^2g_n\|_\infty < +\infty$ , from Theorem 2.8 and Remark 2.9 we infer that the solutions  $w_n$  are equi-Lipschitz in  $[0, +\infty) \times \mathbb{R}$ .

By standard regularity results for parabolic equations, we know that  $w_n$  is also a smooth, classical solution of the problem (4.8) with  $H_n$  in place of  $H$ . Moreover, since

$$A_n(x) (w_n)_{xx} = (w_n)_t + H_n(x, (w_n)_x) \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}, \quad (4.10)$$

$A_n(x) \geq 1/n$ , and  $w_n \in \text{Lip}([0, +\infty) \times \mathbb{R})$ , we infer that  $(w_n)_{xx}$  is bounded on  $[0, +\infty) \times \mathbb{R}$  (with a bound depending on  $n$ ). By differentiating equation (4.10) with respect to  $x$ , we get that the function  $v_n := (w_n)_x$  solves the following Cauchy problem:

$$\begin{cases} (v_n)_t - A_n(x) (v_n)_{xx} + I_n(x, v_n, (v_n)_x) = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ v_n(0, x) = g'_n(x) \geq 0 & \text{in } \mathbb{R}, \end{cases} \quad (4.11)$$

with  $I_n(x, p, \xi) := \partial_x H_n(x, p) + (\partial_p H_n(x, p) - \partial_x A_n(x))\xi$  for  $(x, p, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ . Now notice that  $\partial_x H_n(x, 0) = 0$  for every  $x \in \mathbb{R}$  since  $H_n(\cdot, 0) \equiv 0$  on  $\mathbb{R}$ , in particular  $I_n(x, 0, 0) = 0$ . So we have

$$I_n(x, v_n, (v_n)_x) = \int_0^1 \frac{d}{ds} (I_n(x, sv_n, s(v_n)_x)) ds = b_n(t, x) (v_n)_x + c_n(t, x) v_n$$

with

$$\begin{aligned} b_n(t, x) &:= \int_0^1 \partial_\xi I_n(x, sv_n(t, x), s(v_n)_x(t, x)) ds \\ c_n(t, x) &:= \int_0^1 \partial_p I_n(x, sv_n(t, x), s(v_n)_x(t, x)) ds. \end{aligned}$$

Therefore  $v_n$  is a bounded solution of the following linear homogeneous parabolic equation

$$(v_n)_t - A_n(x) (v_n)_{xx} + b_n(t, x) (v_n)_x + c_n(t, x) v_n = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}$$

with coefficients  $b_n(t, x)$  and  $c_n(t, x)$  that are continuous and bounded in  $(0, +\infty) \times \mathbb{R}$ , in view of (4.9) and of the fact that  $v_n$  and  $(v_n)_x$  are bounded on  $[0, +\infty) \times \mathbb{R}$ . By the classical maximum principle (see, for instance, [18, Ch. 2, Sec. 4, Th. 9]) we conclude that  $(w_n)_x(t, x) = v_n(t, x) \geq 0$  in  $[0, +\infty) \times \mathbb{R}$ , thus, proving the asserted monotonicity of  $w_n$  in  $x$ . Now we pass to the limit in  $n$ : since  $g_n \rightrightarrows_{\text{loc}} g$  and  $A_n \rightrightarrows_{\text{loc}} A$  in  $\mathbb{R}$ ,  $H_n \rightrightarrows_{\text{loc}} H$  in  $\mathbb{R} \times \mathbb{R}$ , and the functions  $w_n$  are equi-Lipschitz in  $[0, +\infty) \times \mathbb{R}$ , we infer that  $w_n \rightrightarrows_{\text{loc}} w$  in  $[0, +\infty) \times \mathbb{R}$ . The desired monotonicity of  $w(t, \cdot)$  follows. The proof of (ii) is analogous.  $\square$

As an easy consequence, we derive the following crucial result. In particular, it implies that condition (L') holds with  $\kappa_r$  independent of  $\omega$ .

**Proposition 4.6.** *For every  $\theta \in \mathbb{R}$ , let us denote by  $u_\theta(t, x, \omega)$  the unique uniformly continuous solution of  $(HJ_1^\omega)$  satisfying  $u_\theta(0, x, \omega) = \theta x$  for  $x \in \mathbb{R}$  and  $\omega \in \Omega$ . The following holds:*

(i) *if  $\theta \geq 0$ , then  $u_\theta$  is also a solution of*

$$u_t - A(x, \omega)u_{xx} + H_+(x, u_x, \omega) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R};$$

(ii) *if  $\theta \leq 0$ , then  $u_\theta$  is also a solution of*

$$u_t - A(x, \omega)u_{xx} + H_-(x, u_x, \omega) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}.$$

Moreover, for every  $\omega$ , the function  $u_\theta(\cdot, \cdot, \omega)$  is Lipschitz continuous on  $[0, +\infty) \times \mathbb{R}$ , with a Lipschitz constant independent of  $\omega$  and locally bounded with respect to  $\theta \in \mathbb{R}^d$ .

*Proof.* Let  $\omega$  be fixed. Items (i) and (ii) follow by applying Proposition 4.5 with  $g(x) := \theta x$  and from the fact that  $H(x, p, \omega) = H_+(x, p, \omega)$  for  $p \geq 0$ ,  $H(x, p, \omega) = H_-(x, p, \omega)$  for  $p \leq 0$ . The asserted Lipschitz estimates for  $u_\theta(\cdot, \cdot, \omega)$  follow from Theorem 2.8 and Remark 2.9.  $\square$

**Remark 4.7.** The Lipschitz properties of  $u_\theta(\cdot, \cdot, \omega)$  stated in the above proposition are independent of the fact that  $H$  is of the form (4.7). Indeed, it suffices that  $H(\cdot, \cdot, \omega)$  belongs to  $\mathcal{H}(\gamma, \alpha_0, \beta_0)$ , with constants  $\gamma > 1$ ,  $\alpha_0, \beta_0 > 0$  independent of  $\omega$ . In particular, if the equation  $(\text{HJ}_\varepsilon^\omega)$  associated to such a  $H$  homogenizes, the effective Hamiltonian  $\overline{H}$  is continuous and coercive, according to Proposition 3.4.

We are now in a position to prove our basic homogenization result.

*Proof of Theorem 4.2.* Let us fix  $\theta \in \mathbb{R}$  and denote by  $u_\theta^\varepsilon(\cdot, \cdot, \omega)$  the unique uniformly continuous solution of  $(\text{HJ}_\varepsilon^\omega)$  satisfying  $u_\theta^\varepsilon(0, x, \omega) = \theta x$  in  $\mathbb{R}$ . Notice that  $u_\theta^\varepsilon$  is identically 0 when  $\theta = 0$ . Let us then assume  $\theta \neq 0$ . Since  $u_\theta^\varepsilon(t, x) = \varepsilon u_\theta(t/\varepsilon, x/\varepsilon, \omega)$ , it is easily seen, in view of Proposition 4.6, that  $u_\theta^\varepsilon(\cdot, \cdot, \omega)$  solves, for every fixed  $\omega$ , the following equation

$$u_t^\varepsilon - \varepsilon A\left(\frac{x}{\varepsilon}, \omega\right) u_{xx}^\varepsilon + H_\pm\left(\frac{x}{\varepsilon}, u_x^\varepsilon, \omega\right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}, \quad (4.12)$$

according to the sign of  $\theta$ . By our assumptions, for each  $\omega \in \Omega_+ \cap \Omega_-$  both equations in (4.12) homogenize for all linear initial data  $g(x) = \theta x$  with coercive and continuous  $\overline{H}_\pm$ , see Remark 4.7. Let us set  $\overline{H}(p) := H_+(p)$  if  $p \geq 0$  and  $\overline{H}(p) := H_-(p)$  if  $p \leq 0$ . The Hamiltonian  $\overline{H}$  is coercive and continuous, since  $\overline{H}_+(0) = \overline{H}_-(0) = 0$ . Moreover, the functions  $u^\varepsilon(\cdot, \cdot, \omega)$  satisfy condition (L'), for a constant  $\kappa_r$  independent of  $\omega$ , in view of Proposition 4.6. By Theorem 3.1 and Proposition 3.4 the asserted homogenization result follows.

To prove the formula for  $\overline{H}$ , we first remark that, since  $\overline{H}(\theta) = \overline{H}_+(\theta)$  for  $\theta \geq 0$  and  $\overline{H}(\theta) = \overline{H}_-(\theta)$  for  $\theta \leq 0$ , we have  $\overline{H}(\theta) \geq \min\{\overline{H}_-(\theta), \overline{H}_+(\theta)\}$ . On the other hand, by comparison, we infer  $u_\theta^\varepsilon \geq u_{\theta_\pm}^\varepsilon$ , where  $u_{\theta_\pm}^\varepsilon$  are respectively solutions of (4.12) with the initial datum  $\theta x$ . From the fact that

$$\overline{H}(\theta) = \lim_{\varepsilon \rightarrow 0^+} -u_\theta^\varepsilon(1, 0, \omega), \quad \overline{H}_\pm(\theta) = \lim_{\varepsilon \rightarrow 0^+} -u_{\theta_\pm}^\varepsilon(1, 0, \omega) \quad \text{a.s. in } \Omega,$$

we also get the opposite inequality.  $\square$

Next, we extend the homogenization result stated in Corollary 4.3 to the case of non-convex Hamiltonians with multiple *pinning points*.

**Definition 4.8.** Let  $H : \Omega \rightarrow C(\mathbb{R}^d \times \mathbb{R}^d)$  be a measurable random field. We shall say that  $H(x, p, \omega)$  is pinned at  $p_0$  if there is a constant  $h_0 \in \mathbb{R}$  such that  $H(\cdot, p_0, \cdot) \equiv h_0$  on  $\mathbb{R} \times \Omega$ .

The following simple observation will be very useful.

**Remark 4.9.** If  $H$  is pinned at  $p_0 \neq 0$ , then  $\tilde{H}(x, p, \omega) := H(x, p + p_0, \omega)$  is pinned at 0. Moreover, the solutions  $u$  and  $\tilde{u}$  of  $(\text{HJ}_1^\omega)$  with Hamiltonians  $H$  and  $\tilde{H}$ , respectively, are related as follows:

$$u(t, x, \omega) = \tilde{u}(t, x, \omega) + p_0 x.$$

This implies that  $(\text{HJ}_\varepsilon^\omega)$  with Hamiltonian  $H$  homogenizes if and only if  $(\text{HJ}_\varepsilon^\omega)$  with Hamiltonian  $\tilde{H}$  homogenizes.

We are now ready to prove our final homogenization result.

**Theorem 4.10.** *Let  $H : \Omega \rightarrow \mathbb{C}(\mathbb{R} \times \mathbb{R})$  be a stationary random field satisfying  $H(\cdot, \cdot, \omega) \in \mathcal{H}(\gamma, \alpha_0, \beta_0)$  for every  $\omega$ , with constants  $\gamma > 1$ ,  $\alpha_0, \beta_0 > 0$  independent of  $\omega$ . Let us furthermore assume that*

- (i)  *$H$  is pinned at  $p_1 < p_2 < \dots < p_n$ ;*
- (ii)  *$H(x, \cdot, \omega)$  is convex (or level-set convex if  $A \equiv 0$ ) on each of the intervals  $(-\infty, p_1)$ ,  $(p_1, p_2)$ ,  $\dots$ ,  $(p_n, +\infty)$ , for every  $(x, \omega) \in \mathbb{R} \times \Omega$ .*

*Then there exists a set  $\hat{\Omega}$  of full measure in  $\Omega$  such that, for every  $\omega \in \hat{\Omega}$ , equation  $(\text{HJ}_\varepsilon^\omega)$  homogenizes for all  $g \in \text{UC}(\mathbb{R})$ .*

*Proof.* Let us assume, to fix ideas, that  $H(x, \cdot, \omega)$  is convex on each of the intervals  $(-\infty, p_1)$ ,  $(p_1, p_2)$ ,  $\dots$ ,  $(p_n, +\infty)$ , for every  $(x, \omega) \in \mathbb{R} \times \Omega$ . It follows from the hypotheses that  $H$  can be written as

$$H(x, p, \omega) = \min\{H_i(x, p, \omega) : 1 \leq i \leq n+1\} = \begin{cases} H_1(x, p, \omega) & \text{if } p \leq p_1; \\ H_2(x, p, \omega) & \text{if } p_1 \leq p \leq p_2; \\ \dots & \dots \\ H_{n+1}(x, p, \omega) & \text{if } p \geq p_n, \end{cases}$$

where  $H_1, \dots, H_{n+1} : \Omega \rightarrow \mathbb{C}(\mathbb{R} \times \mathbb{R})$  are stationary random fields such that, for every fixed  $\omega$ , each  $H_i(\cdot, \cdot, \omega)$  is a convex Hamiltonian belonging to  $\mathcal{H}(\gamma, \tilde{\alpha}_0, \tilde{\beta}_0)$ , with same  $\gamma$  and possibly different constants  $\tilde{\alpha}_0, \tilde{\beta}_0 > 0$ , independent of  $\omega$  and  $i \in \{1, \dots, n+1\}$ .

The proof is by induction on  $n$ . The case  $n = 1$  follows from Corollary 4.3 and Remark 4.9. Let us now assume that the assertion holds for  $n-1$  and prove it for  $n$ . To this aim, notice that  $H$  can be written as

$$H(x, p, \omega) = \min\{H_-(x, p, \omega), H_+(x, p, \omega)\} = \begin{cases} H_-(x, p, \omega) & \text{if } p \leq p_n; \\ H_+(x, p, \omega) & \text{if } p \geq p_n, \end{cases}$$

with  $H_+ := H_{n+1}$  and  $H_-$  defined as

$$H_-(x, p, \omega) := \begin{cases} H_1(x, p, \omega) & \text{if } p \leq p_1; \\ H_2(x, p, \omega) & \text{if } p_1 \leq p \leq p_2; \\ \dots & \dots \\ H_n(x, p, \omega) & \text{if } p \geq p_{n-1}. \end{cases}$$

By the inductive hypothesis, the assertion holds both for  $H_-$  and  $H_+$ . Another application of Remark 4.9 together with Theorem 4.10 yields that the assertion holds for  $H$  as well. The case when  $A \equiv 0$  and  $H$  is level-set convex in  $p$  on each of the intervals  $(-\infty, p_1)$ ,  $(p_1, p_2)$ ,  $\dots$ ,  $(p_n, +\infty)$  can be handled similarly.  $\square$

**Remark 4.11.** Let  $H, H_1, \dots, H_{n+1}$  be as above. By the same reasoning as in the proof of Theorem 4.2 and by induction, it can be easily shown that the effective Hamiltonian  $\overline{H}$  associated to  $H$  satisfies

$$\overline{H}(\theta) = \min\{\overline{H}_1(\theta), \dots, \overline{H}_{n+1}(\theta)\} \quad \text{for every } \theta \in \mathbb{R},$$

where  $\overline{H}_i$  is the effective Hamiltonian associated to  $H_i$ , for each  $i \in \{1, \dots, n+1\}$ .



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